be shown that there exists one nontrivial closed form exact solution, namely, for $\alpha = 1.0$ and $\beta = 1.0$, it follows that

$$u_e = u_{e_c}(x/x_c) \tag{12a}$$

$$F = (\eta^2/2) - 6(1 - e^{-\eta^2/4}) \tag{12b}$$

and

$$F'/\eta = 1 - 3e^{-\eta^2/4} \tag{12c}$$

References

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Interaction of an Oscillating Magnetic Field with Fluid in Couette Flow

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THE problem of Couette flow interaction with an oscillating transverse magnetic field is solved by means of a transformation that demonstrates the similarity between a set of transient MHD systems and a set of transient heat-conduction systems. Consider two concentric nonconducting cylinders with the inner cylinder at rest. The electrodes, situated at both ends of the annulus, are short-circuited by means of an external connection. For convenience, the distance between the cylinders is assumed small compared to either of the radii. Curvature effects are thus neglected, and the problem is solved in the Cartesian coordinate system. Thus, nonconducting stationary and moving plates are located at $y = 0$ and $y = h$, respectively. A sinusoidal magnetic field B_y is applied perpendicular to the plate. An electrically conducting fluid is in laminar flow, with velocity $u(y,t)$, between the plates. All physical quantities will be considered, at most, functions of y and t.

The interaction of the velocity field $\mathbf{u}$ with the magnetic field $\mathbf{B}$ produces an electric field $\mathbf{E}$, which is given in direction and magnitude by the vector relation $\mathbf{E} = \mathbf{u} \times \mathbf{B}$. The induced current is given by Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, where σ is the electrical conductivity. The magnitude and direction of the Lorentz body force in the momentum equation is given by $\mathbf{J} \times \mathbf{B}$. The magnitude and direction of the electric field vary with the same frequency of the magnetic field. However, as indicated by the double cross-product, the Lorentz force opposes the fluid motion throughout the total cycle of the magnetic field; the frequency associated with the electric field is twice that of the magnetic field. For convenience, the magnetic Reynolds number is taken to be much less than unity. It is interesting to note that this force system, composed of the moving plate and the magnetic field, is quite unsymmetric as far as the fluid is concerned. That is, every particle of moving fluid senses the magnetic force instantaneously (a sort of democratic force), whereas, in the instant the magnetic field is off, the particles of fluid sense the force associated with the moving plate by means of their neighbors, which are between them and the moving plate (an aristocratic force). Strictly speaking, the first mechanism propagates with the velocity of light, whereas the second mechanism propagates with the rate of diffusion of vorticity, which depends on viscosity.

The equation of momentum in the direction of flow is

$$\partial u/\partial t = \alpha(\partial^2 u/\partial y^2) - \alpha N_H^2\, u \sin^2\omega t \tag{1}$$

where

u = velocity
t = time
α ≡ $\mu/\rho h^2$
μ = fluid viscosity
ρ = fluid density
h = distance between plates
y = dimensionless distance from the bottom plate
N_H = Hartmann number
ω = frequency associated with the magnetic field

The boundary and initial conditions are

$$u(0, t) = 0 \qquad u(1, t) = 1 \qquad u(y, 0) = y \tag{2}$$

The limiting form of the solution to Eqs. (1) and (2) when $\omega \to 0$ is obtained by using the transformation $\tau = \omega t$ and neglecting the acceleration term

$$\underset{\omega\to 0}{\mathcal{L}}\ u(y, t) = \frac{\sinh(N_H \sin\omega\, t)y}{\sinh(N_H \sin\omega\, t)} \tag{3}$$

The velocity profile then oscillates between two steady states with a frequency 2ω, responding instantaneously to the changing magnetic field.

The general solution to Eqs. (1) and (2) may be obtained by means of the transformation

$$u(y, t) = V(y, t) \exp - \beta t + \frac{\beta \sin 2\,\omega t}{2\omega} \tag{4}$$

where $\beta \equiv \alpha\, N_H^2/2$. Equations (1) and (2) then reduce to

$$\frac{\partial V}{\partial t} = \alpha \frac{\partial^2 V}{\partial y^2} \tag{5}$$

$$\left.\begin{aligned} V(0, t) &= 0 \\ V(l, t) &= \exp \beta t - \frac{\beta \sin 2\,\omega t}{2\omega} \\ V(y, 0) &= y \end{aligned}\right\} \tag{6}$$

The foregoing system also describes heat conduction between two infinite parallel plates between which a linear temperature exists initially, and, at $t = 0$, the temperature on one boundary is made to vary exponentially in time. The solution to Eqs. (4–6) is[1]

$$u(y, t) = 2 \sum_{n=1}^{\infty} (-1)^{n-1} \sin n\pi\, y \times \exp\left[- \alpha\left(n^2\pi^2 + \frac{N_H^2}{2}\right)t + \frac{\beta}{2}\sin 2\,\omega t\right] \left\{\frac{1}{n\pi} + \alpha\, n\pi \times \int_0^t \exp\left[- \alpha\left(n^2\pi^2 + \frac{N_H^2}{2}\right)t' - \frac{\beta}{2}\sin 2\,\omega t'\right] dt' \right\} \tag{7}$$

When ω is large so that $\exp(\beta/2\omega)\sin 2\,\omega t$ is negligible, Eq. (7) can be easily integrated:

$$u(y, t) = \frac{\sinh(2^{1/2}/2)\, N_H y}{\sinh(2^{1/2}/2)\, N_H} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin n\, \pi y\, (N_H^2/2) \exp - \alpha[n^2\pi^2 + (N_H^2/2)]t}{(n\pi)[n^2\pi^2 + (N_H^2/2)]} \tag{8}$$

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Equation (8) is the solution to the MHD Couette flow formation from system starting at rest. The solution to this problem was solved by Tao.[2] However, Eq. (8) is much more compact and lends itself much more easily to physical interpretation and calculation than does the solution presented by Tao. Equation (8) shows that for large ω the fluid does not have time enough to respond fully to the changing magnetic field and responds only to the rms value.

A sample calculation was performed for the following conditions:

$$\begin{aligned} \sigma &= 25 \text{ mho/m} \\ h &= 0.1 \text{ m} \\ \mu &= 1 \text{ cp} \\ B &= 6350 \text{ gauss} \\ N_H &= 10 \\ \omega &= 0.1 \text{ cps} \end{aligned}$$

After 500 sec, the system will reach an almost steady state that, for all practicality, is nonoscillating in the velocity field and is uniformly oscillating in the current field $\mathbf{J} = \sigma \mathbf{u} \times \mathbf{B}$. The actual current distortion is less than 5% of the peak value, thus indicating that a value of $\omega = 0.1$ cps is still "large." It is interesting to note that the distortion of the a.c. field is minimized for both $\omega \to 0$ and $\omega \to \infty$. When the fluid velocity field partially follows (lags behind) the magnetic field, distortion in the a.c. field arises.

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Inner-Outer Expansion Method for Couple-Stress Effects

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COUPLE stresses are associated with couples per unit area. For materials in which they are present, the usual stress tensor is not symmetric. Couple-stress effects in linear elasticity have been studied by Mindlin and Tiersten,[1] Mindlin,[2] and Sadowsky, Hsu, and Hussain.[3] These authors noted that, generally, the main effects of couple stresses are to be found in thin layers adjacent to the boundary curves of the problem. For example, Mindlin's exact solution for a stress-free circular hole in a plate subjected to a uniform stress at infinity clearly exhibits this boundary-layer-type behavior.

The pertinent parameter for couple-stress effects is $\epsilon = l/a$ where l is the couple-stress material constant and a is a characteristic length of the problem. The linear couple-stress elastic theory is valid for $\epsilon < 1$.[1] It appeared to the first author that couple-stress problems with small ϵ can be treated quite generally as a singular perturbation problem by following the formalism of the method of inner-outer expansion. For general references to this method, see Van Dyke.[4] In essence, we take advantage of the fact that ϵ is small and proceed to construct an approximate solution that remains uniformly valid in the limit $\epsilon \to 0$. This method can readily be applied to couple-stress problems with arbitrary geometries and boundary conditions. The generality of the method should compensate more than adequately for the loss of accuracy. For example, the elliptic-hole version of Mindlin's problem could be very simply treated, whereas an attempt on an exact solution proved to be a formidable task.[5] The present authors have developed the general inner-outer expansion solution including the leading terms for couple-stresses for a hole of arbitrary shape. These results will appear in a later publication. In the present note, we shall illustrate the method of inner-outer expansion by applying it to the simple case of a circular hole in a plate under uniform tension at infinity.

From Mindlin,[2] the potential functions φ and ψ satisfies the following equations:

$$\nabla^4 \varphi = 0 \tag{1}$$

$$\nabla^2\psi - \epsilon^2 \nabla^4 \psi = 0 \tag{2}$$

$$\frac{\partial}{\partial r}(\psi - \epsilon^2 \nabla^2 \psi) = -2(1-\nu)\epsilon^2 \frac{1}{r}\frac{\partial}{\partial \theta}\nabla^2 \varphi \tag{3}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(\psi - \epsilon^2 \nabla^2 \psi) = 2(1-\nu)\epsilon^2 \frac{\partial}{\partial r}\nabla^2 \varphi \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{5}$$

and all variables have been properly nondimensionalized. Note that Eqs. (1–4) are not independent, for Eqs. (1) and (2) can be derived from Eqs. (3) and (4). The usual stress components and the couple-stress components are derived from the potentials by means of

$$\left.\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \\
\sigma_\theta &= \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \psi}{\partial \theta} \\
\tau_{r\theta} &= -\frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} - \frac{1}{r}\frac{\partial \psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} \\
\tau_{\theta r} &= -\frac{1}{r}\frac{\partial^2 \varphi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} + \frac{\partial^2 \psi}{\partial r^2} \\
\mu_r &= \frac{\partial \psi}{\partial r} \\
\mu_\theta &= \frac{1}{r}\frac{\partial \psi}{\partial \theta}
\end{aligned}\right\} \tag{6}$$

where $\tau_{\theta r}$ is the shear stress in the r direction on the θ-constant surface and similarly for $\tau_{r\theta}$. μ_r is the couple-stress in z direction on the r constant surface and similarly for μ_θ. The boundary conditions at infinity are

$$\sigma_x = 1 \quad \text{at} \quad r \to \infty \tag{7}$$

and all other stresses and couple-stresses vanish. The boundary conditions on the boundary of the hole are

$$\sigma_r = \tau_{r\theta} = \mu_r = 0 \quad \text{at} \quad r = 1 \tag{8}$$

We begin by saying that the solution for φ and ψ can be expanded in the following form:

$$\begin{aligned} \varphi &= \Phi_0(\xi, \theta) + \epsilon\Phi_1(\xi, \theta) + \epsilon^2\Phi_2(\xi, \theta) + \ldots \\ \psi &= \Psi_0(\xi, \theta) + \epsilon\Psi_1(\xi, \theta) + \epsilon^2\Psi_2(\xi, \theta) + \ldots \end{aligned} \tag{9}$$

where $\xi = r - 1$. The preceding expansions are called the

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